

The Fermi Statistics of a Weakly Excited Column of Granular Particles in a Vibrating Bed

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Abstract

A one dimensional experiment in granular dynamics is carried out to test the thermodynamic theory of weakly excited granular systems [Hayakawa and Hong, Phys. Rev. Lett. 78, 2764(1997)] where granular particles are treated as spinless Fermions. The density profile is measured and then fit to the Fermi distribution function, from which the global temperature of the system, T , is determined. Then the center of mass, $\langle z(T) \rangle$, and its fluctuations, $\langle \Delta z(T)^2 \rangle$, are measured and plotted as functions of T . The Fermi function fits the density profile fairly well, with the value of T being reasonably close to the predicted value. The scaling behavior of $\langle z(T) \rangle$ and $\langle \Delta z(T)^2 \rangle$ is in excellent agreement with the theory.

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1 Introduction

This paper is an experimental test of the theory presented and tested in preceding papers [1] and [2]. The thermodynamic theory of Hayakawa and Hong(HH) presented in [2] was tested in [1] with extensive Molecular Dynamics simulations. The purpose of this paper is to test the theory of HH experimentally for a one dimensional vibrating granular system. A one dimensional system is peculiar in the sense that randomness associated with collisions is suppressed in contrast with what happens in higher dimensions. Nevertheless, the Fermi statistics arising from hard core repulsion still apply to this simple one dimensional system. As in [1], we determine the configurational statistics of a one dimensional vibrating granular system by properly taking the ensemble average of the steady state. Then we measure physical quantities of the system and compare them to those predicted by HH. First, we measure the density profile and determine the dimensionless Fermi temperature,

$$T = \frac{T_f}{mgD},$$

where T_f is the Fermi temperature, by fitting the density profile to the Fermi function. Second, we compare the measured Fermi temperature T to those predicted by

the theory in [1] and [2]. Third, we measure the center of mass $\langle z(T) \rangle$ and the fluctuations of the center of mass $\langle \Delta z(T) \rangle$ for the vibrating bed and test the scaling predictions of [2]. We will first summarize the theory presented by HH.

2 Background of Fermi Statistics and Thermodynamic Theory of Granular Materials

The system being studied here is a dense, dissipative, nonequilibrium, granular system, where the mean free path of the grains is of the order of a few particle diameters. Hence, each particle may be considered to be effectively confined in a cage as in the free volume theory of a dense liquid [3]. In such a case, an observation has been made in [4] that the basic granular state is not a gas, but a solid or crystal, and thus, the effective thermodynamic theory based on the free energy argument may be more appropriate than the kinetic theory in studying this state. In such a case, the configurational statistics of the steady state may be determined by the variational method as the most probable or minimum free energy state.

To be more specific, consider the excitation of disordered granular materials confined in a box with vibrations of the bottom plate. The vibrations will inject energy into the system which cause the ground state to become unstable, and a newly excited state will emerge with an expanded volume. The time averaged configurational statistics of this new excited state have undergone structural distortions. However, the degree of distortions from the ground state may be small for a weakly excited state, possibly justifying the use of an effective thermodynamic theory based on the variational principle. Such a thermodynamic approach may be further justified by the following two experiments conducted previously.

1. Weakly or moderately excited regime: Clement and Rajchenbach(CR) [4] have performed an experiment with the vibrational strength, Γ , of the order one for a two dimensional vibrating bed, using inclined side walls to suppress convections. Here, $\Gamma = A\omega^2/g$ with A and ω , the amplitude and frequency of the vibrating plane, and g , the gravitational constant. CR have found that the ensemble-averaged density profile as a function of height from the bottom layer obeys a universal function that is independent of the phase of oscillations of the vibrating plate. Namely, it is independent of the kinetics imposed on the system. One conceptually important point here is that the reference point of the density profile is not the bottom plate, but the bottom layer, which of course is fluidized.

2. Highly excited regime: Warr and Hansen(WH) [5] have performed an experiment on highly agitated, vertically vibrating beds of $\Gamma \approx 30 - 50$ using steel balls with a small coefficient of restitution. They have found that the collective behaviors of this vibrated granular medium *in a stationary nonequilibrium state* exhibits strong similarities to those of an atomistic fluid in *thermal equilibrium* at the corresponding particle packing fraction, in particular, in the two-point correlation function [6,7].

The results of both experiments indicate that for both moderate or highly excited systems, one-to-one correspondence seems to exist between *configurational* statistics of the nonequilibrium stationary state and the equilibrium thermal state. In fact, this is not so surprising considering that upon vibration, the granular materials expand and increase the volume of the system. In turn, this increase corresponds to a rise in the potential energy after the configurational average is appropriately taken. Then the problem reduces to the packing problem, and the temperature-like variable, T ,

can be associated to the vibrating bed. The existence of distinctive configurational statistics in the density profile of CR (and also in WH in a special case) appears to be fairly convincing evidence that kinetic aspects of the excited granular materials may be separated out from the statistical configurations. These observations are the basis of the thermodynamic theory proposed in [2]. Note that the Fermi statistics is essentially the macroscopic manifestation of the classical excluded volume effect and the anisotropy which causes the ordering of potential energy by gravity. The top surface of the granules plays the role of a Fermi surface, and the thin boundary layers that appear near the top layer upon excitation play the role of excited electrons of the Fermi gas in metals.

3 Thermodynamic Theory of Weakly Excited Granular Systems

1. Fermi temperature: In [2], the vibrating system was viewed from two different points of view. One may view it as a mechanical system, in which case the expansion is due to excitation induced by mechanical vibrations of strength $\Gamma = A\omega^2/g$. In this case, the expansion is purely due to kinetics. In an attempt to develop a thermodynamic theory, such a system was also viewed [2] as a thermal system in contact with a heat reservoir. Therefore a global temperature T was associated with the granular system. In this case, the expansion is purely a thermal expansion. By equating the thermal expansion, Δh , defined as the increase in the center of mass, and the kinetic expansion, $gH_o(\Gamma)/\omega^2$, where $H_o(\Gamma)$ is the dimensionless jump height of a single ball [2,8] on the vibrating plate(see Eq.(1) in [2]), a closure in the thermodynamic theory of powders was obtained in [2]. Since the density decrease above the Fermi surface is not sharp, but smooth, one may replace $gH_o(\Gamma)/\omega^2$ with $h_o(\Gamma)/\alpha$, where $h_o(\Gamma)$ is the maximum jump height of a single ball at the Fermi surface (or vibrating plate) determined by MD simulations. The factor α was introduced to both expressions to incorporate (i) the smooth decrease in the density profile near the Fermi surface, and (ii) the suppression in the jump height due to dissipation. By equating the kinetic expansion and the thermal expansion, one obtains the following explicit relationships between the temperature T and the control parameters:

$$\frac{T}{mg} = \frac{1}{\pi} \sqrt{\frac{6D[gH_o(\Gamma)/\omega^2]}{\alpha}}, \quad (1a)$$

$$\frac{T}{mg} = \frac{1}{\pi} \sqrt{\frac{6D[h_o(\Gamma)]}{\alpha}}. \quad (1b)$$

Note that when a single particle is on a vibrating plate, the energy from the bottom wall is transferred to the particle through direct contact. In the case of many particles, the supplied energy at the vibrating plate must first travel through other particles locked in their respective lattice states before reaching those particles in the fluidized layer. With this in mind, α can be thought of as a dissipation constant. In [1], α was empirically determined to have a value of $\alpha = 64/5$. This value of α fit the data in [1], regardless of the type of vibration used.

2. The center of mass and its fluctuations: Since the density profile is given by the Fermi function, it is straightforward to compute the center of mass, $\langle z(T) \rangle$, and its fluctuations, $\langle (\Delta z)^2 \rangle$. This leads to the following equations:

$$\Delta z(T) = z(T) - z(0) = \frac{D\mu_o\pi^2}{6}\left(\frac{T}{mgD\mu_o}\right)^2, \quad (2a)$$

$$\langle (\Delta z)^2 \rangle = \langle (z(T) - \langle z \rangle)^2 \rangle = \frac{\langle (\Delta h)^2 \rangle}{\mu_o^2} = \frac{\pi^2}{3}\left(\frac{T}{mgD}\right)^3 \frac{D^2}{\mu_o^2}. \quad (2b)$$

Note that the total expansion, $\Delta h(T) \equiv \mu_o \Delta z$, and its fluctuations $\langle (\Delta h)^2 \rangle / D^2 = \langle \mu_o (\Delta z)^2 / D^2 \rangle$ are only a function of the dimensionless Fermi temperature $T_f = T/mgD$ as expected. Further, note that Eq.(2b) is an indirect confirmation that the specific heat is linear in T as it is for the non-interacting Fermi gas.

4 The One Dimensional Experiment

We can now explain the experimental setup in some detail. The one dimensional column of particles used in this experiment was 30 plastic beads. The beads, having holes through their centers, were strung through a thin piece of copper wire. The wire was stretched extremely tight between two horizontal rods clamped to a ring stand. Each of the beads could freely move up and down the copper wire. The beads were all close to the same size, with an average diameter of $D = 4.24 \text{ mm}$. Differences in the diameters of the beads were not noticeable to the naked eye. The bottom of the copper wire ran through a plastic plate connected directly to the mechanical vibrator, which is very similar to an audio speaker. One of the plastic beads was glued to the plastic plate, acting as the bottom wall when the vibration was turned on. A schematic diagram of the set up is shown in Fig.1. The mechanical vibrator was then connected to a function generator, which allowed one to control the type, amplitude, and frequency of the mechanical vibration. For this experiment, the only type of vibration used was a sine wave, $A \sin(\omega t)$, where A is the amplitude and $\omega = 2\pi f$ is the angular frequency. The frequency of the vibration and the voltage which set the amplitude were both controlled with the function generator. The relationship between the voltage and the vibration amplitude is nonlinear and quite complex. Hence, instead of deriving the mathematical formula, for a given applied voltage, we directly measured the corresponding amplitude of the vibration of the plate. We find that voltage with a range of $1.00V$ to $10.00V$ corresponds to an amplitude of 0.495 mm to 1.890 mm . In this experiment, the frequency f was kept at $f = 40Hz$ while the voltage was varied by $0.5V$ increments from $1.00 - 10.00V$. The amplitudes, A , and the corresponding vibrational strength ($\Gamma = A\omega^2/g$) with g being the gravitational acceleration are listed in Table 1. This set of control parameters satisfies the criterion of the weakly excited granular system used to test the theory of HH, namely that

$$R = \frac{\mu D}{\Gamma A}, \quad (3)$$

with μ being the number of layers. To analyze the system, pictures were taken with a high speed digital camera while the beads were subjected to vibration. The camera was placed on a stand in front of the beads, while pictures were taken of the system in motion. This is schematically illustrated in Fig.2. A set of 10 pictures was taken for each value of Γ used in this experiment, as well as a picture of the beads at rest in the beginning and end of each run. Then, all the images were analyzed to obtain the position of the center of each bead for each image. Once the individual bead coordinates were known, they were fed into a computer program to calculate the average density profile

of the system, the average center of mass, and the fluctuations of the center of mass for each voltage setting.

To find the density profile, equally sized boxes with a height equal to the average diameter of a bead were constructed so that each box essentially contained, one bead when the system was at rest in the ground state. Note that the zero reference point in all our measurements is not the vibrating plate directly, but rather a single particle glued to the bottom. This was done to ensure that all collisions were the same for all particles, including the bottom wall collision.

Before proceeding, we want to clarify two points regarding the density measurement. First, the maximum square packed density for a one dimensional system of spheres is calculated to be $\pi/6$. With our experiment, we used pictures of the system to analyze the density. Since the pictures are two dimensional, we can treat our column of spherical beads as column of disks confined to one dimensional motion. Using this analysis, the maximum square packed density for our column of disks is calculated to be $\pi/4$. However, on the density plots presented in this paper, the density exceeded this value for certain data points. This is because box sizes were assumed to all have a length and width equal to the average bead radius. In reality, however, this was not true, leading at times to a density greater than $\pi/4$. Second, since the bottom layer is the reference point, one may expect that the density of the first data point should remain the same. However, we noticed that the values of the density of the bottom changed slightly in our data. This was due to the erosion of the balls that occurred due to the repeated motion on the copper wire. This erosion of the center hole caused the configuration of the bottom few balls to become slightly more dense as more trials were completed.

For this experiment, 50 boxes were chosen because the motion of the column of beads never reached that height, even for the largest voltage. Therefore, every bead was included in the calculations of all the density profiles. Since this is a one dimensional experiment, each bead was considered to be a circle instead of a sphere. Hence, the maximum possible area of a bead in a box, a_o , is πr^2 where

$$r = \langle D \rangle / 2 = \sum_i D_i / 2N$$

is the average radius of a bead. D_i is the diameter of bead i , and N is the total number of beads. In the ground state, when the system is at rest, each box up to the Fermi surface contains one bead with an area of a_o . However, when the beads are subjected to vibration, the system expands, and the time averaged position of each grain rises. The area of the grains in each box is computed from the pictures, which is denoted as a . Then, the density of each box, having an approximate value between 0 and 1, was computed using

$$\rho = \frac{a}{a_o}. \quad (4)$$

This process was done for all 50 boxes and in all 10 pictures for each voltage used. An average density for each box was then found by summing up the 10 densities per box and dividing by 10. The density profile for a particular voltage was obtained by plotting the average density of each box against the position of the center of that particular box. The center of the first box is chosen as the origin. The profile is then fit with the Fermi profile, and a global temperature, T , is obtained. The fitting of the density profile was done with a non-linear least squares program. Computer programs were used to obtain the average center of mass and its fluctuations. The center of mass is computed using the following expression:

$$z = \frac{\sum_i z_i m_i}{\sum_i m_i}, \quad (5)$$

where z_i is the vertical position in centimeters of the bead i and m_i is the mass in grams of the bead i . These summations were carried out over all 30 beads. This formula was used to get the center of mass for each picture, and then the average was taken over all 10 pictures to obtain the average center of mass, $\langle z \rangle$.

The average fluctuations of the center of mass are computed using the standard definition:

$$\langle (\Delta z)^2 \rangle = \frac{[\sum_j (z(j) - \langle z \rangle)]^2}{10}, \quad (6)$$

where $z(j)$ is the center of mass of an individual picture, and $\langle z \rangle$ is the average center of mass found previously for a particular value of voltage. In this case, the summation is carried out over 10, the number of pictures taken per run. Both the average center of mass and the fluctuations were computed for each voltage setting.

5 Data and Results

Density Profile: This experiment was carried out for 30 particles using a sine wave vibration ranging from 1.00 – 10.00V and varying by increments of 0.5V while being held at a constant frequency of $f = 40Hz$. The change in V is synonymous with a change in the vibrational strength, Γ . The density profile, ρ , as a function of the vertical position, z , for the different voltages was fit with the Fermi profile as shown in Fig.(3). Note that for an electron gas, the Fermi density profile is given by

$$\rho(\epsilon_i) = \frac{1}{1 + \exp[(\epsilon_i - \bar{\mu})/T_f]}, \quad (7)$$

where ϵ_i is the energy level of electron i , $\bar{\mu}$ is the Fermi energy as a function of T , and T_f is the Fermi temperature. As discussed in [1], for granular systems, $\bar{\mu}(T)$ is independent of the temperature because the density of the state is constant. We denote this constant as $\mu_o \equiv \bar{\mu}(T = 0)$, which is the initial number of layers. Note that the dimensionless Fermi energy for our granular system at $T = 0$ is the initial number of layers, while the Fermi energy $\bar{\mu}$, is replaced with $mgD\mu(T)$. For the configurational statistics of granular materials in a vibrating bed, the energy level is given by the average position of the particle. Thus, the energy level is given by the gravitational energy, $\epsilon_i = mgz_i \equiv mgD\bar{z}$, where z_i is the vertical position of particle i and \bar{z} is the dimensionless position measured in units of D . Note that \bar{z} , runs from 0 to 50 because the position of the first grain is taken as the origin. The relationship between the dimensionless fitting Fermi temperature T in this paper and the Fermi temperature T_f in Eq.(2a) is given by $T = T_f/mgD$. This leads to the following density profile,

$$\rho(z) = \frac{1}{1 + \exp[(\epsilon_i(\bar{z}) - \bar{\mu})/T_f]} \equiv \frac{1}{1 + \exp[(\bar{z} - \mu)/T]} \quad (8)$$

An example of Eq.(8) fit to our data can be seen in Fig.3, which is for a particular voltage of 10V. Note that the Fermi statistics are valid for all of our data according to the condition that $R \gg 1$ as stated in Eq.(3). In our experiment, R changes from 76.3 to 5.6 as shown in Table 2.

Fermi temperature fitting: When fitting the experimental data with the Fermi function, the following two adjustable parameters are used: the Fermi energy, μ , and

the global temperature T as defined previously. The parameter μ shifts the location of the Fermi energy horizontally, while the temperature, T , controls the curvature around the Fermi energy. As discussed above, the Fermi energy is expected to remain constant regardless of the temperature T if the density of states is indeed constant. However, the experimental fitting shows that there is some slight variation in μ . In this experiment, μ varies from 29.01 to 30.57, a relative increase of about $\Delta\mu/\mu = 0.051$. The scaling behavior of the center of mass and its fluctuations don't seem to be affected by the change in μ . The fitting values obtained for μ and T are also listed in Table 2.

When the temperature, T , obtained by the Fermi fitting is compared to the theoretical prediction from Eq.(1a), it seems to match fairly well. As discussed in [1], a best fit value of $\alpha = 64/5$ was used to obtain the theoretical temperature. Table 3 shows the values of the fitting temperature T , and two theoretical temperatures $T_{theory1}$ and $T_{theory2}$. Both are obtained by using Eq.(1a). The difference in the two predicted values is because of differences in the dissipation constant α . For $T_{theory1}$, the value of $\alpha = 64/5$ was used, which was the value found empirically using MD simulations in [1]. This value of α provided predicted values all within 30% of the measured T . However, it is not necessarily correct to assume that energy will be dissipated through our experimental grains exactly the same way as it was in the MD simulations of [1]. Depending on how the constants in the MD program are set, the particle reactions during a collision could be quite different than that which occurred in our experiment. Hence, $T_{theory2}$ was obtained with a best fit value of $\alpha = 8$, determined empirically from our data. With this new value of α , most of the predicted values are well within 10% of what we measured.

The Center of Mass: The relative increase in the center of mass of the one dimensional column is denoted as $\langle \Delta z(T) \rangle$, which is the difference in the actual position of the average center of mass, $\langle z(T) \rangle$, and that of the ground state $\langle z(T=0) \rangle$. Using our definitions for undimensional variables, Eq.(2a) becomes

$$\langle \Delta z(T) \rangle = \frac{\pi^2 D}{6\mu} T^2. \quad (9)$$

Note the appearance of the factor D in Eq(9). This is because, in Eq.(2a), the temperature T has dimensions of length, while here it is dimensionless. The center of mass is plotted in Fig.4 as a function of T^2 for different values of Γ . By using the solid line as a guide for the eye, one can see that the graph seems to confirm the scaling predictions of the Fermi statistics, $\Delta z \propto T^2$. There is, however, a discrepancy in the proportionality constant C . Using the average value of μ calculated from Table 2, theory predicts that one should get $C = \pi^2 D/6\mu \approx 0.000234$. The experimental results yield, 0.0001, a difference of about a factor of 2.5 when compared to the theory. It has been shown in [1] that this discrepancy is due to the extreme sensitivity of the center of mass to the Fermi energy μ . When the density of states is independent of the energy, the Fermi energy must remain constant. However, in this one dimensional experiment, it changes ever so slightly for each value of Γ . This small change does not seem to change the Fermi fitting of our density profiles, but it does affect the amplitude of the scaling relationships.

Fluctuations of the center of mass: The fluctuation of the center of mass, $\langle (\Delta z)^2 \rangle$, is given by Eq.(2b), which becomes

$$\langle (\Delta z)^2 \rangle = \frac{\pi^2}{3} \frac{D^2}{\mu^2} T^3, \quad (10)$$

when changing to dimensionless variables. The fluctuations of the center of mass were

also measured and plotted as a function of T^3 in Fig.5. Using the guide line, the graph seems to confirm the validity of the Fermi statistics which imply that $\langle (\Delta z)^2 \rangle \propto T^3$. Once again, the proportionality constant deviates from the theoretical value of 6.67×10^{-8} . From the graph, the slope is approximately 1.00×10^{-6} . Considering that the amplitude of the center of mass is off by a factor of about 2.5, one should expect that the fluctuations would be off as well. This is, again, due to the sensitivity of the Fermi integral to μ , where small changes in μ are greatly magnified in the fluctuations, resulting in the deviation from the theoretical constant. A similar trend was found using MD simulations[1]. Another source of error is due to the fact that the fluctuations in the center of mass are quite large in the vibrating column, and all the particles, not just those near the surface, fluctuate in the continuum space of the experiment. This is different from the Fermi model which makes all particles below the Fermi surface, inactive. So the positions of the grains, on average, may obey the Fermi distribution function well, but the magnitude of the fluctuations may not. It is very suprising, though, that the T^3 prediction of the Fermi statistics still seems to hold.

6 Conclusion

We now summarize the main results of this experiment as follows. First, the configurational statistics of grains in a one dimensional system subject to vibration seem to obey the Fermi statistics of spinless particles for weakly excited systems as was predicted[1,2]. Second, the temperatures determined by fitting the Fermi profile are fairly close to the theoretical predictions. Third, the scaling relations of the center of mass and its fluctuations obey the Fermi statistics, but there are discrepancies in the amplitudes or proportionality constants. Note that another source of error in this experiment results from friction. The beads were confined to vibration in a single column by the copper wire that ran through their centers. After many trials, filings from the inside of the plastic beads were found at the base of the setup. The motion of the beads on the wire eroded away some of the hole in the center of each bead. This friction could be responsible for some of the error in the results. Nevertheless, we find that the essence of the Fermi statistics survives in this one dimensional experiment.

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Table Captions

Table 1: The values of the vibrational amplitude A , and the vibrational strength Γ for each corresponding voltage V .

Table 2: A set of parameters obtained in the experiment for various voltages. Here, μ , is the dimensionless Fermi energy, T is the dimensionless Fermi temperature, and $R = \mu D / \Gamma A$ is the parameter that determines the validity of the Fermi statistics.

Table 3: The comparison of the experimentally measured Fermi Temperature, T , and two theoretical Fermi temperatures determined from Eq.(1a). The first, $T_{theory1}$, is determined using the value of $\alpha = 64/5$, as was determined in [1]. The second theoretical temperature, $T_{theory2}$ is determined by using a best fit $\alpha = 8$ value, determined empirically from the data.

Figure Captions

Figure 1: An illustration of the experimental setup used to vibrate a column of beads at a controlled amplitude and frequency.

Figure 2: An illustration of the experimental setup used to take pictures of the vibrating column of beads with a high speed digital camera.

Figure 3: The density profile of our one dimensional granular system vibrating under $10V$, which is equivalent to $\Gamma = 12.18$. The profile is fit with the Fermi density function, giving values for the undimensionalized variables $\mu = 30.57$ and $T = 4.57$. One can see the profile fits extremely well.

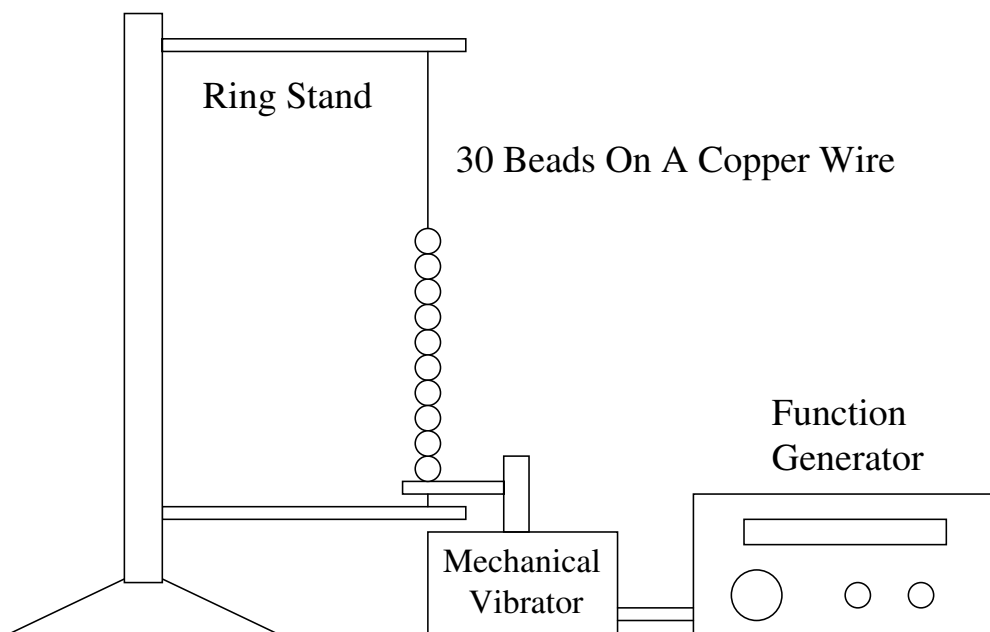
Figure 4: Plots of the average center of mass $\langle z_{avg} \rangle$ as a function of T^2 . The data is shown with the circles, while dashed line is the best fit line. The data matches the theoretical prediction of a T^2 dependence for the average center of mass.

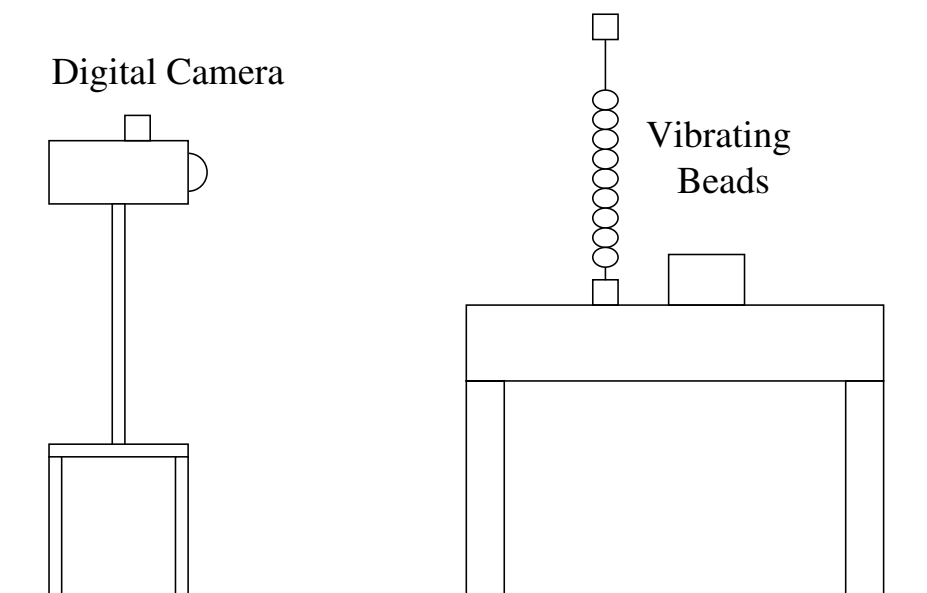
Figure 5: Plots of the fluctuations of the average center of mass $\langle (\Delta z)^2 \rangle$ as a function of T^3 . The data is shown with the circles, while the dashed line is the best fit line. The data follows the theoretical prediction of a T^3 dependence for the fluctuations.

Voltage	Amplitude	Γ
1.5V	0.50 mm	3.22
2.0V	0.55 mm	3.54
2.5V	0.64 mm	4.12
3.0V	0.71 mm	4.57
3.5V	0.76 mm	4.89
4.0V	0.78 mm	5.02
4.5V	0.86 mm	5.53
5.0V	0.92 mm	5.92
5.5V	0.96 mm	6.18
6.0V	1.05 mm	6.76
7.0V	1.20 mm	7.72
7.5V	1.30 mm	8.37
8.0V	1.40 mm	9.02
8.5V	1.56 mm	10.05
9.0V	1.65 mm	10.64
9.5V	1.80 mm	11.60
10.0V	1.89 mm	12.18

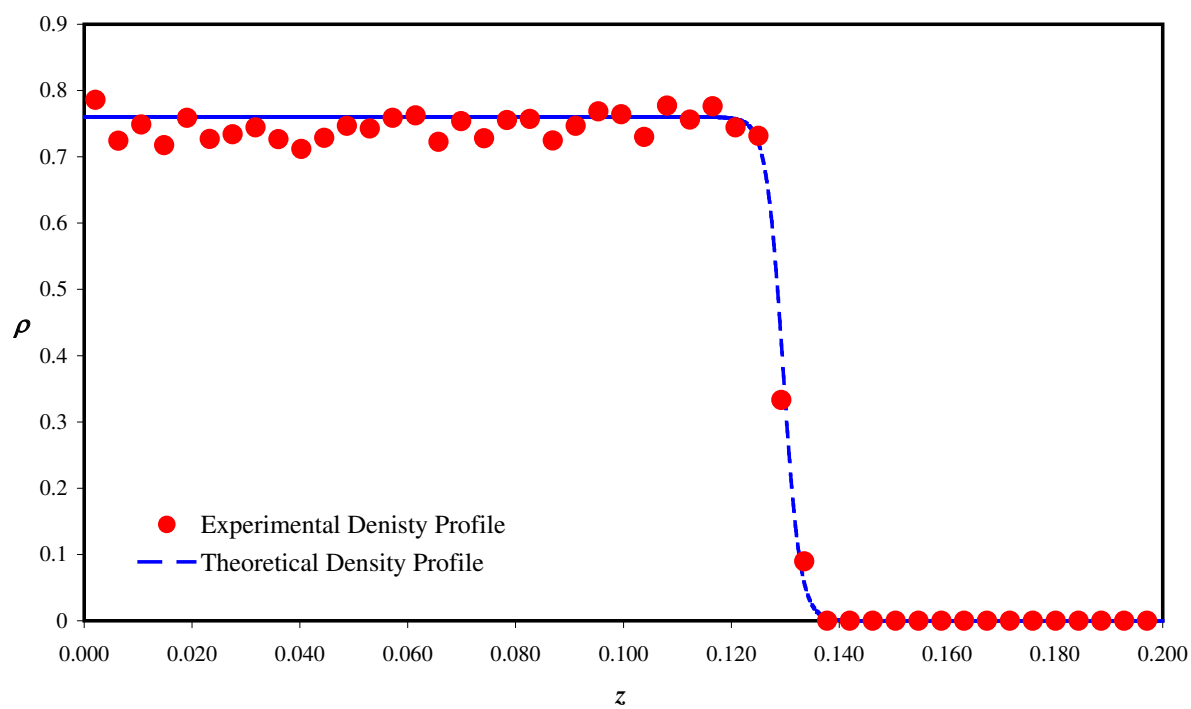
Voltage	Γ	μ	T	$R = \frac{\mu D}{\Gamma A}$
1.50V	3.22	29.01	1.16	76.3
2.00V	3.54	29.01	1.75	63.1
2.50V	4.12	29.25	2.10	50.0
3.00V	4.57	29.36	2.62	38.3
3.50V	4.89	29.41	2.71	33.5
4.00V	5.02	29.43	2.91	31.8
4.50V	5.53	29.60	3.06	26.3
5.00V	5.92	29.72	3.12	23.1
5.50V	6.18	29.72	3.35	21.2
6.00V	6.76	29.72	3.49	17.7
7.00V	7.72	30.07	3.90	13.7
7.50V	8.37	30.19	3.99	11.8
8.00V	9.02	30.19	4.22	10.1
8.50V	10.05	30.19	4.31	8.2
9.00V	10.64	30.31	4.43	7.3
9.50V	11.60	30.42	4.49	6.2
10.00V	12.18	30.57	4.57	5.6

Voltage	T	$T_{theory1}$	$T_{theory2}$
1.50V	1.16	1.48	1.87
2.00V	1.75	1.62	2.05
2.50V	2.10	1.86	2.35
3.00V	2.62	2.03	2.57
3.50V	2.71	2.14	2.71
4.00V	2.91	2.19	2.77
4.50V	3.06	2.36	2.98
5.00V	3.12	2.48	3.14
5.50V	3.35	2.56	3.24
6.00V	3.49	2.73	3.46
7.00V	3.90	3.00	3.79
7.50V	3.99	3.16	4.00
8.00V	4.22	3.32	4.20
8.50V	4.31	3.55	4.49
9.00V	4.43	3.69	4.66
9.50V	4.49	3.90	4.93
10.00V	4.57	4.00	5.06

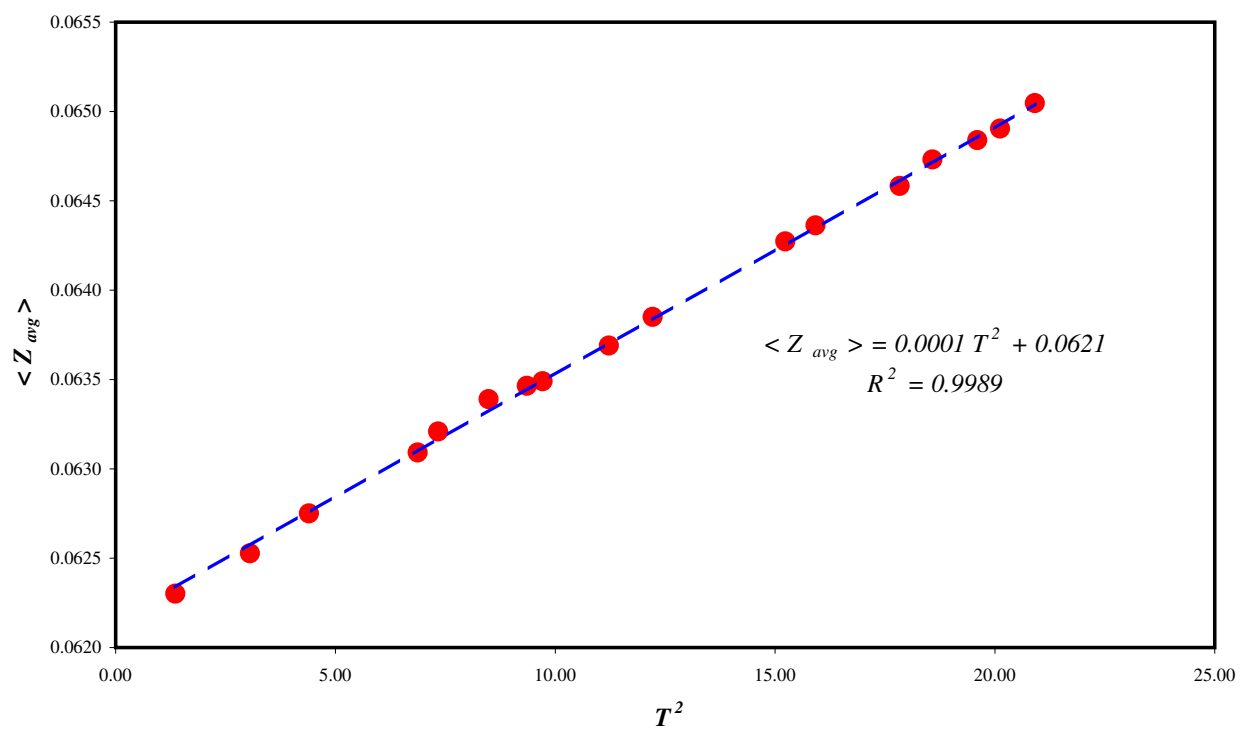




Density (ρ) vs. Height (z)



$\langle Z_{avg} \rangle$ vs. T^2



$\langle (\Delta Z)^2 \rangle$ vs. T^3

